P661 Homework # 3 09/09/04

Due Tuesday 9/21/04

Read Chapter 2. Homework problems in this chapter will be given in the next homework set.

The Jacobi Theta functions $\Theta_i(z,\tau)$ are a canonical set of natural analytic functions of the torus. Here we are focusing on $\Theta_i(\tau) = \Theta_i(0,\tau)$, where $\Theta_1(\tau) = 0$ and $(q = e^{2\pi i\tau})$:

$$\begin{aligned}
\Theta_2(\tau) &= 2q^{1/8}\Pi_{n=1}^{\infty}(1+q^n)^2(1-q^n) = \sum_{n\in\mathbf{Z}} q^{(n+1/2)^2/2} \\
\Theta_3(\tau) &= \Pi_{n=1}^{\infty}(1+q^{n-1/2})^2(1-q^n) = \sum_{n\in\mathbf{Z}} q^{n^2/2} \\
\Theta_4(\tau) &= \Pi_{n=1}^{\infty}(1-q^{n-1/2})^2(1-q^n) = \sum_{n\in\mathbf{Z}} (-1)^n q^{n^2/2} \\
\eta(\tau) &= q^{1/24}\Pi_{n=1}^{\infty}(1-q^n) = \sum_{n\in\mathbf{Z}} (-1)^n q^{3(n-1/6)^2/2}
\end{aligned} \tag{1}$$

where $\Theta_1'(0,\tau) = 2\pi\eta^3(\tau)$. They satisfy the following identities :

$$2\eta^3 = \Theta_2\Theta_3\Theta_4 \tag{2}$$

$$\Theta_3^4 = \Theta_2^4 + \Theta_4^4 \tag{3}$$

Part 1

(1) Relate the partition function of the superstring theory to the above Theta functions. (This reproduces what was done in class. You may consult Chapter 10 and other parts in the textbook.)

(2) Find $\eta(\tau+1)$. Write $\Theta_i(\tau+1)$ in terms of $\Theta_j(\tau)$.

Part 2:

(3) Convince yourself of the above Jacobi identity (3) (e.g., in the small q expansion, check the coefficients of the lowest terms).

(4) Prove the identity (2).

Hint: using the infinite product expressions, show that

$$f^2(q) = \frac{\Theta_2\Theta_3\Theta_4}{2\eta^3} = 1$$

via showing $f(q) = f(q^2)$ and $f(0) = 1$.

(5) Derive the Poisson resummation formula:

$$\sum_n e^{-\pi c_1 n^2 + c_2 n} = \frac{1}{\sqrt{c_1}} \sum_k e^{-\frac{\pi}{c_1}(k + c_2/2\pi i)^2} \tag{4}$$

by using the identity

$$\sum_n \delta(x - n) = \sum_k e^{2\pi i k x}$$

and by integrating it over $e^{-\pi c_1 x^2 + c_2 x}$.

Use the Poisson resummation formula (with appropriate choices of the constants c_1 and c_2) to find $\Theta_3(-1/\tau)$, $\Theta_2(-1/\tau)$ and $\Theta_4(-1/\tau)$ in terms of $\Theta_i(\tau)$.

Use Eq.(2) to show that

$$\eta(-1/\tau) = w\sqrt{-i\tau}\,\eta(\tau) \tag{5}$$

where $w^3 = 1$. Argue that the cube root of unity should be $w = 1$ (hint: consider pure positive imaginary τ, i.e., $\tau_1 = 0$ and $\tau_2 > 0$).